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A catastrophe associated with the misuse of Fisher's solution for dispersion on a sphere

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Abstract. Perrin has considered the direction $\boldsymbol{l}$ of a line in a body which is subject to a large number of elementary rotations, and obtained an exact expression for the resulting probability distribution of $\boldsymbol{l}$. Fisher has developed a theory for the statistical analysis of samples of $\boldsymbol{l}$ on the assumption that the populations of the $\boldsymbol{l}$ have a certain assumed probability distribution. It has since been concluded elswhere in the literature that, since Fisher's distribution is very similar to that of Perrin, Fisher's distribution (being apparently simpler than Perrin's) might well be suitable even when Perrin's is known to have a sounder physical basis. We show that this is a false conclusion in at least one situation when the asymptotic form (for large variance of $\boldsymbol{l}$) is of physical significance. Fisher's distribution would predict that zero electrical resistivity is produced by multiple elastic small-angle electron scattering. On the other hand, Perrin's distribution (exact for this situation) leads to a finite resistivity and a simple exponential decay of current when the electric field is removed.

1. Introduction

Perrin (1928) gave the first exact solution to the problem of determining the probability distribution of the position on the surface of a sphere of a particle which diffuses on the surface of that sphere, starting from some fixed point. The problem is formally equivalent to the angular probability distribution of a fixed direction in a body which has suffered many random elementary rotations; or to the angular probability distribution of the direction of motion of a particle which has suffered many random elementary elastic deflections; or to the angular distribution of waves produced when an incident plane wave has travelled through a medium which causes random multiple small-angle elastic scattering.

If θ is the angular displacement, Perrin's result $P_{\mathrm{P}}(\theta, t)$ defined so that $P_{\mathrm{P}}(\theta, t)\,\mathrm{d}\theta$ is the probability that the direction lies between θ and $\theta+\mathrm{d}\theta$ at time t, is given by

$$P_{\mathrm{P}}(\theta, t) = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \exp[-n(n+1)Dt]\mathrm{P}_n(\cos\theta)\sin\theta \qquad (1)$$

where $\mathrm{P}_n(\cos\theta)$ is a Legendre polynomial, and D is the diffusion coefficient. We note that (1) has axial symmetry and that

$$P_{\mathrm{P}}(\theta, 0) = \delta(\theta). \qquad (2)$$

Fisher (1952) proposed an angular distribution $P_{\mathrm{F}}(\theta, t)$ in order to develop a theory for the statistical analyses of such measured quantities as the angular distribution of rock magnetism:

$$P_F(\theta, t) = \frac{\kappa}{2\sinh\kappa}\exp(\kappa\cos\theta)\sin\theta \tag{3}$$

where κ (which would depend on t in a time evolving situation) is a measure of the variance of θ.

Provided that the angular distribution cannot be definitely attributed to a physical mechanism equivalent to diffusion, there is no *a priori* reason for preferring $P_P(\theta, t)$ to $P_F(\theta, t)$. Indeed since $P_F(\theta, t)$ is simpler in form and has mathematical properties which make it more useful for statistical work, one would generally prefer to assume $P_F(\theta, t)$.

It is trivial to show that as $\kappa \to \infty$, $P_F(\theta, t)$ is confined to the region close to $\theta = 0$ and tends to the *plane* two-dimensional gaussian distribution:

$$P_F(\theta, t) \underset{\kappa\to\infty}{\to} P_G(\theta, t) = \frac{2\theta}{\theta_0^2}\exp\left(-\frac{\theta^2}{\theta_0^2}\right) \tag{4}$$

where $\theta_0^2 = 2/\kappa$. As well as the correct limiting behaviour given in (4), Fisher adduced further confirmation for the physical significance of his distribution from a satisfactory comparison of (3) with measurements on rock magnetism.

Roberts and Ursell (1960) obtained Perrin's result (1) as the limiting form of the solution to the problem of the random walk (*finite* step length) on a sphere. They also showed that

$$Dt = \tfrac{1}{4}\theta_0^2 \tag{5}$$

where θ_0^2 is the variance of θ which would have occurred if the diffusion had taken place on a plane. Indeed, as one would expect, $P_P(\theta, t)$, like $P_F(\theta, t)$ tends to $P_G(\theta, t)$ as t tends to zero. For as t tends to zero, $P_P(\theta, t)$ is only significant around $\theta = 0$ so we may expand $\mathrm{P}_n(\cos\theta)$ as a power series

$$\mathrm{P}_n(\cos\theta) = 1 - \tfrac{1}{4}(n^2+n)\theta^2 + \ldots,$$

and, because all the higher-order Legendre polynomials become equally important in the limit, may replace the summation in (1) by an integral. The result

$$P_P(\theta, t) \underset{t\to 0}{\to} P_G(\theta, t)$$

then follows directly.

Since $P_P(\theta, t)$ and $P_F(\theta, t)$ both tend to $P_G(\theta, t)$ as t tends to zero, and since $P_P(\theta, t)$ and $P_F(\theta, t)$ both tend to a uniform distribution as t and $1/\kappa$ respectively tend to infinity, Roberts and Ursell concluded that $P_F(\theta, t)$ would probably be a satisfactory distribution to use mathematically even when the physics dictates the true distribution to be $P_P(\theta, t)$. Numerical calculations of the *absolute* differences between $P_F(\theta, t)$ and $P_P(\theta, t)$ confirmed them in their opinion.

Therefore, on account of the apparently greater simplicity of $P_F(\theta, t)$ it was suggested by Roberts and Ursell (and reiterated by Breitenberger 1963) that for practical purposes one may use $P_F(\theta, t)$ with confidence.

It is our purpose to show that in one particular application, where $P_P(\theta, t)$ should be a proper description of the physical situation, the use of $P_F(\theta, t)$ is an unacceptable approximation. The application we refer to is the problem of calculating the equilibrium forward current in a stream of particles which move in a region where there is (i) a uniform accelerating field and (ii) a scattering mechanism which imparts to the particles very small random elastic deflections. The expression for the equilibrium forward current, which is derived in the next section, depends on the distribution of directions which would be produced after a certain time in an initially parallel beam by the small elastic deflections, when the accelerating field is not present. We have thus set up a scattering mechanism designed specifically to give an angular distribution equivalent to that produced by the random walk on a sphere with very small step. The solution to this problem is $P_P(\theta, t)$.

We should point out that the formal problem described is probably very close to the actual situation when electrons are scattered by dislocations in a metal, for we assert that dislocations scatter electrons elastically through very small angles. The elasticity of the scattering is expected because a dislocation is a static lattice defect with effectively infinite mass so far as an electron is concerned. As for the very small angles of scattering involved, they result from the long range of the strain field of a dislocation compared with the wavelength of an electron on the Fermi surface; experimental confirmation is provided by Terwilliger and Higgins (1973).

We shall show in the next section that if a steady electric field were applied to the dislocated metal, the use of $P_F(\theta, t)$ would predict an electric current which would increase indefinitely with time (the catastrophe referred to in our title); but the use of $P_P(\theta, t)$ would predict a finite equilibrium current, a true exponential decay of current when the field is switched off and therefore a well defined relaxation time. Thus the apparently complicated form of $P_P(\theta, t)$ leads to the simplest of all behaviours for the forward current.

2. Forward current decay from multiple small-angle scattering

We consider an initially parallel beam of electrons travelling through a medium in which they are continually deflected through very small angles in random directions. If there is no applied electric field to maintain the original direction of motion, the beam will progressively spread out, and the forward current will decay.

The angular distribution of directions of motion after a time t is given exactly by (1) and approximately by (3). We define the forward current in the beam to be

$$F(0, t) = \int_0^\pi I(\theta, t) \cos\theta \, \mathrm{d}\theta \tag{6}$$

where $I(\theta, t)$ is the number of electrons travelling at time t at angle θ to the initial direction $\theta = 0$, and $\cos\theta$ picks out the forward component of electric current. Now $I(\theta, t)$ is given exactly by $P_P(\theta, t)$ or by $P_F(\theta, t)$, provided Fisher's approximation is valid. Therefore, taking $F(0, 0)$ to be unity, we have

$$F_F(0, t) = \int_0^\pi P_F(\theta, t) \cos\theta \, \mathrm{d}\theta \tag{7}$$

whence it may be easily shown that

$$F_F(0, t) = \coth\left(\frac{1}{2Dt}\right) - 2Dt \tag{8}$$

where we have used (4) and (5) to write κ in terms of Dt. Similarly, using the exact expression $P_P(\theta, t)$ we obtain

$$F_P(0, t) = \exp(-2Dt). \tag{9}$$

Obtaining (9) is simple if one notices that $\cos\theta$ is $P_1(\cos\theta)$ and remembers the orthogonal properties of Legendre polynomials.

We already see that (9), the exact form of the forward current decay, is more simple than the approximate form (8); but now let us calculate the equilibrium electric current produced by a steady electric field E. If E were switched on at time $\tau = -\infty$ then the forward current $J(0, t)$ flowing at time $\tau = t$ is given by

$$J(0, t) = j\int_{-\infty}^{t} F(0, t-\tau)\,\mathrm{d}\tau \tag{10}$$

where $j\,\mathrm{d}\tau$ is the element of current created by E in the time interval between τ and $\tau+\mathrm{d}\tau$ (the constant of proportionality j depends on the dynamical properties of an electron), and $F(0, t-\tau)$ is the factor by which $j\,\mathrm{d}\tau$ has decayed at time $\tau = t$. Changing the variable in (10) by putting $T = t-\tau$ we may write down two alternative expressions for $J(0, t)$

$$J_F(0, t) = j\int_0^{\infty}\left[\coth\left(\frac{1}{2DT}\right) - 2DT\right]\mathrm{d}T \tag{11}$$

and

$$J_P(0, t) = j\int_0^{\infty}\exp(-2DT)\,\mathrm{d}T. \tag{12}$$

It is evident that $J_F(0, t)$ diverges, that is to say a steady field does not produce a finite equilibrium current, whereas $J_P(0, t)$ is equal to $j/2D$.

The reason why Fisher's distribution, though differing only slightly from Perrin's in absolute magnitude, should predict an infinite current must arise from the fact that $P_F(\theta, t)$ does not approach the uniform distribution rapidly enough as t becomes large.

We compare the asymptotic forms of $P_F(\theta, t)$ and $P_P(\theta, t)$:

$$P_F(\theta, t) \rightarrow \frac{1}{2}\left[1+\left(\frac{1}{2Dt}\right)\cos\theta\right]\sin\theta \tag{13}$$

and

$$P_P(\theta, t) \rightarrow \tfrac{1}{2}[1+3\exp(-2Dt)\cos\theta]\sin\theta \tag{14}$$

as $t \rightarrow \infty$. Thus we see that $P_F(\theta, t)$ tends to a uniform distribution as $1/t$, whereas $P_P(\theta, t)$ goes like $\exp(-t)$.

Therefore although $P_F(\theta, t)$ and $P_P(\theta, t)$ have very little absolute difference, the slower rate at which $P_F(\theta, t)$ tends to the uniform distribution may lead to serious errors from its use in describing phenomena where the behaviour of $P(\theta, t)$ at large t is important. We should also draw attention to the simple exponential form (9) of the exact solution for decay of the forward current when the electric field is switched off. This exact solution leads to a unique definition of a relaxation time, which is given by $1/2D$ in our notation.

Concerning Fisher's distribution we draw the following conclusion: contrary to certain conclusions in the literature, the use of Fisher's distribution might be expected to lead to serious errors in any calculation where (i) the true distribution has arisen from a number of elementary (angular) steps and (ii) the asymptotic form of the distribution is important. However, we should emphasise that our conclusion is not a criticism of Fisher's distribution when used for the purpose of statistical analysis as he intended. For although (i) may often be satisfied approximately, it is most unlikely that (ii) would be satisfied in statistical analyses.

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